American Scientific Research Journal for Engineering, Technology, and Sciences (ASRJETS)
ISSN (Print) 2313-4410, ISSN (Online) 2313-4402

http://asrjetsjournal.org/

On Chromatic Number and Edge-Chromatic Number of the Ottomar Graph

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Abstract

The path graph P_n, consists of the vertex set $V = \{1,2,\dots,n\}$ and the edge set $E = \{\{1,2\},\{2,3\},\dots,\{n-1,n\}\}$. The cycle graph C_n, is the path graph, P_n with an additional edge $\{1,n\}$. Define the Ottomar Graph, denoted by $O_{n,m}$, to be the graph $C_n, n \in \mathbb{Z}^+, n \geq 3$, with a vertex connected by a path P_2 to a vertex of $C_m, m \in \mathbb{Z}^+, n \geq 3$. C_n is called the heart while C_m is called a foot (feet for plural). Note that there are n copies of C_m. The chromatic number of a graph G, denoted by $\chi(G)$, is the minimum number of colors the vertices of G maybe colored such that any two adjacent vertices have different colors. The edge-chromatic number of a graph G, denoted by $\chi_e(G)$, is the minimum number of colors the edges of G maybe colored such that any two incident edges have different colors. The chromatic number and the edge-chromatic number of the ottomar graph are determined. When will the two invariants be equal or when will they be unequal? When the connecting path P_k has order greater than 2, what happens to the value of $\chi(G)$ and $\chi_e(G)$? Also in the paper, the other coloring invariants are compared and investigated with chromatic number and edge-chromatic number.

Keywords: path; cycle; chromatic number; edge-chromatic number; ottomar graph; generalized ottomar graph.

1. Introduction

A pair $G = (V, E)$ with $E \subseteq E(V)$ is called a graph (on V). The elements of V are the vertices of G, and those of E the edges of G.

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The vertex set of a graph G is denoted by V_G and its edge set by E_G. Therefore $G = (V_G, E_G)$. The *path graph* P_n, consists of the vertex set $V = \{1, 2, \ldots, n\}$ and the edge set $E = \{\{1,2\}, \{2,3\}, \ldots, \{n-1, n\}\}$. The cycle graph C_n, is the path graph, P_n with an additional edge $\{1, n\}$. The chromatic number of a graph G, denoted by $\chi(G)$, is the minimum number of colors the vertices of G maybe colored such that any two adjacent vertices have different colors. The edge-chromatic number of a graph G, denoted by $\chi_e(G)$, is the minimum number of colors the edges of G maybe colored such that any two incident edges have different colors.

Known Result 1 [2] If C_n is a cycle of order n, then

$$\chi(C_n) = \begin{cases} 2 & if\ n\ is\ even \\ 3 & if\ n\ is\ odd \end{cases} \tag{1.1}$$

and,

$$\chi_e(C_n) = \begin{cases} 2 & if\ n\ is\ even \\ 3 & if\ n\ is\ odd \end{cases} \tag{1.2}$$

Define the Ottomar Graph, denoted by $O_{n,m}$, to be the graph $C_n, n \in \mathbb{Z}^+, n \geq 3$, with a vertex connected by a path P_2 to a vertex of $C_m, m \in \mathbb{Z}^+, n \geq 3$. C_n is called the *heart* while C_m is called a *foot* (*feet* for plural). Note that there are n copies of C_m.

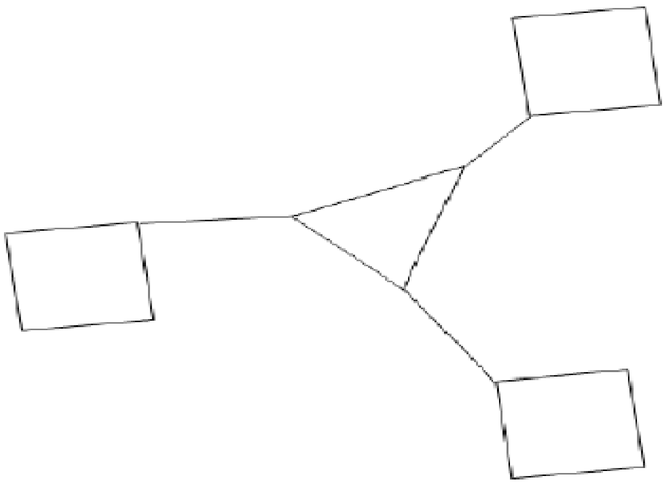

Figure 1: Ottomar Graph: $O_{3,4}$

2. Identities of Chromatic Number and Edge-Chromatic Number of Ottomar Graph

Theorem 2.1

For all integers $m, n \geq 3, \chi(O_{n,m}) = 3$, if:

- *Case 1: m, n are both odd*
- *Case 2: m is odd, n is even*
- *Case 3: m is even, n is odd*

and

- $\chi(O_{n,m}) = 2$ *if m, n are even.*

Proof:

- Case 1: <u>m and n are both odd</u>

If m, n are odd, then by equation (1.1), $\chi(C_m) = \chi(C_m) = 3$, thus $\chi(O_{n,m}) \neq 2$. Suppose the vertices of C_m and C_n are colored with the same set of different colors say a_1, a_2, a_3. Then the path P_2 is attached to a vertex colored say $a_i, i = 1, 2, 3$ of C_m and the other end vertex is attached to a vertex colored say a_j , $j = 1,2,3$ of C_n, where $a_i \neq a_j$. Hence, three is the minimum number of colors to color the vertices of $O_{n,m}$, where m, n are both odd. Consequently, $\chi(O_{n,m}) = 3$.

- Case 2: <u>m is odd, n is even</u>

If m is odd and n is even, then by equation (1.1), $\chi(C_m) = 3$ and $\chi(C_n) = 2$, so $\chi(O_{n,m}) \neq 2$. Suppose that the vertices of C_n are colored with two of the colors that also color the vertices of C_m, say a_1, a_2 for C_n and a_1, a_2, a_3 for C_m. Then a path P_2 is attached to vertex colored say $a_i, i = 1,2,3$ of C_m and other end vertex is attached to a vertex colored $a_j, j = 1,2$ of C_n, where $a_i \neq a_j$. This means that the minimum number of colors to color the vertices of $O_{n,m}$, where n is even and m is odd is three. Thus, $\chi(O_{n,m}) = 3$.

- Case 3: <u>m is even, n is odd</u>

Proof of this case is similar to case 2.

- Case 4: <u>m, n are both even</u>

If m, n are even then by equation (1.1), $\chi(C_m) = \chi(C_n) = 2$. Suppose the vertices of C_m and C_n are colored with the same set of different colors, say a_1, a_2. Then a path P_2 is attached to a vertex colored say $a_i, i = 1,2$ of

C_m and the other end vertex is attached to a vertex colored say $a_j, j = 1{,}2$ of C_n, where $a_i \neq a_j$. Hence, two is the minimum number of colors to color vertices of $O_{n,m}$, where, n, m are both even. Consequently, $\chi(O_{n,m}) = 2$. ■

Theorem 2.2 *For all integers* m, n, $m, n \geq 3$, $\chi_e(O_{n,m}) = 3$.

Proof:

To prove this theorem, we consider the following cases:

- Case 1: <u>m, n are both odd</u>

If m, n are both odd, then by equation (1.2), $\chi_e(C_m) = \chi_e(C_n) = 3$. Thus, $\chi_e(O_{n,m}) \neq 2$ since there exist three incident edges. Suppose the edges of C_m and C_n are colored with the same set of different colors, say b_i, b_j, b_k. Then a path P_2 connecting C_m and C_n must have an edge colored with one of the colors b_1, b_2, b_3, say $b_i, i = 1{,}2{,}3$, where b_i is incident to edges colored b_j and b_k of C_m and is also incident to edges colored b_j and b_k of C_n, $b_i \neq b_j \neq b_k$. Hence, three is the minimum number of colors to color the edges of $O_{n,m}$, where n, m are both odd. Consequently, $\chi_e(O_{n,m}) = 3$.

- <u>Case 2: m, n are both even</u>

If m, n are both even, then by equation (1.2), $\chi_e(C_m) = \chi_e(C_n) = 2$. Thus, $\chi_e(O_{n,m}) \neq 2$ since there exist three incident edges. Without a loss of generality, suppose the edges of C_m and C_n are colored with same set of different colors b_1, b_2. Then a path P_2 connecting C_m and C_n where its edge is incident to colors b_1, b_2 edges of C_m and C_n, must have an edge colored with b_3 such that $b_1 \neq b_2 \neq b_3$. Hence, three is the minimum number of colors to color the edges of $O_{n,m}$, where m, n are both even. Consequently, $\chi_e(O_{n,m}) = 3$.

- Case 3: <u>m is odd, n is even</u>

If m is odd, n is even, then by equation (1.2), $\chi_e(C_m) = 3$ and $\chi_e(C_n) = 2$. Thus $\chi_e(O_{n,m}) \neq 2$. Without a loss of generalization, suppose that the edges of C_n are colored with two of the colors that also color the edges of C_m, say b_1, b_2 colors of C_n and b_1, b_2, b_3 colors of C_m. Then a path P_2 connecting C_m and C_n must have an edge colored with b_3 and must also be incident to edges colored b_1 and b_2 of C_m and must also be incident to edges colored b_1 and b_2 of C_n. Hence, three is the minimum number of colors to color the edges of $O_{n,m}$, where m is odd, n is even. Consequently, $\chi_e(O_{n,m}) = 3$.

- Case 4: <u>m is even, n is odd</u>

If m is even, n is odd, then the proof of this case is similar to case 3. ■

Corollary 2.1

For all integers $m, n \geq 3$

$$\chi(O_{n,m}) \leq \chi_e(O_{n,m})$$

Proof:

Note that for the cases where m, n are both odd, m is odd, n is even, and m is even, n is odd, by Theorem 2.1, $\chi(O_{n,m}) = 3$ and by Theorem 2.2, $\chi_e(O_{n,m}) = 3$. Thus, $\chi(O_{n,m}) \leq \chi_e(O_{n,m})$. Similarly, for cases where m, n are both even, by Theorem 2.1, $\chi(O_{n,m}) = 2$ and by Theorem 2.2, $\chi_e(O_{n,m}) = 3$. Thus, $\chi(O_{n,m}) \leq \chi_e(O_{n,m})$. Therefore, in all cases, $\chi(O_{n,m}) \leq \chi_e(O_{n,m})$. ∎

Remark 2.1 For all integers $k \geq 3$, $\chi(P_k) = \chi_e(P_k) = 2$.

3. Generalized Ottomar Graph

Define the Generalized Ottomar Graph, ${O^k}_{n,m}$, is graph $C_n, n \in \mathbb{Z}^+, n \geq 3$, with each vertex connected by a path $P_k, k \in \mathbb{Z}^+, k \geq 3$ to a vertex of C_m, $m \in \mathbb{Z}^+, m \geq 3$. C_n is called a *heart* while C_m is called a *foot* (*feet* for plural). Note that there are n copies of C_m.

Theorem 3.1 *For all integers* $k = 3, 4$, $\chi({O^k}_{n,m}) = 3$ *if,*

- m, n *are both odd*
- m *is odd,* n *is even*
- m *is even,* n *is odd*

and

$$\chi({O^k}_{n,m}) = 2, \quad if\ m, n\ are\ both\ even.$$

Proof:

- Case 1: m, n are both odd

If m, n are both odd, then by equation (1.1), $\chi(C_m) = \chi(C_n) = 3$. Thus, $\chi({O^k}_{n,m}) \geq 3$. Suppose the vertices of C_m and C_n are colored with the same set of different colors say a_1, a_2, a_3. Consider the following subcases where $k = 3$ (odd) and $k = 4$ (even):

- subcase 1.1: If $k = 3$

Then a path P_3 with vertices colored with two from the same set of different colors a_1, a_2, a_3, is attached to a vertex colored say $a_i, i = 1,2,3$ of C_m and the other end vertex is also connected to a_i of C_n, such that the second (middle) vertex of P_3 is $a_j, j = 1,2,3$, where $a_i \neq a_j$. Thus three is the minimum number of colors to color the vertices of $O^3{}_{n,m}$, where m, n are both odd. Consequently, $\chi\left(O^3{}_{n,m}\right) = 3$.

- <u>subcase 1.2:</u> If $k = 4$

Note that by Remark 2.1, $\chi(P_4) = 2$. Suppose further that P_4 is colored with two from the same set of different colors that color the vertices of C_m and C_n, say $a_i, a_j, i, j = 1,2,3$. Then, the first vertex of P_4 is colored a_i of C_m and the last vertex of P_4, colored a_j is attached to a_j of C_n. The other vertices of P_4 are colored a_i, a_j such that no two adjacent vertices have the same color. Thus, three is the minimum number of colors to color the vertices of $O^4{}_{n,m}$, where m, n are both odd. Consequently, $\chi\left(O^4{}_{n,m}\right) = 3$.

- Case 2: m is odd, n is even

If m is odd, n is even, then by equation (1.1), $\chi(C_m) = 3$ and $\chi(C_n) = 2$. Thus, $\chi\left(O^k{}_{n,m}\right) \geq 3$. Suppose that the vertices of C_n are colored with two of the different colors that also color the vertices of C_m, say a_i, a_2 colors for C_n and a_1, a_2, a_3 colors for C_m. Consider the following subcases where $k = 3$ (odd) and $k = 4$ (even):

- <u>subcase 2.1:</u> If $k = 3$

Then a path P_3 is attached to vertex colored say $a_i, i = 1\,, 2\,, 3$ of C_m and the other end vertex is also connected to a vertex colored a_i of C_n, such that the second (middle) vertex of P_3 is $a_j, j = 1\,, 2\,, 3$, where $a_i \neq a_j$. Thus, three is the minimum number of colors to color the vertices of $O^3{}_{n,m}$ where m is odd and n is even. Consequently, $\chi\left(O^3{}_{n,m}\right) = 3$.

- <u>subcase 2.2:</u> If $k = 4$

Note that by Remark 2.1, $\chi(P_4) = 2$. Suppose further that P_4 is colored with two from the same set of different colors that color the vertices of C_m and C_n, say a_1, a_2. Then, the first vertex of P_4 is attached to a vertex colored $a_i, i = 1,2$ of C_m and is adjacent to a vertex colored $a_j, j = 1,2$, which is the second vertex of P_4, and the last vertex is then connected to a vertex colored $a_j, j = 1,2$ of C_n, where $a_i \neq a_j$. Note that the vertices of C_m are colored $\{a_1, a_2, a_3\}$. Thus, three is the minimum number of colors to color the vertices of $O^4{}_{n,m}$, where m is odd, n is even. Consequently, $\chi\left(O^4{}_{n,m}\right) = 3$.

- Case 3: m is even, n is odd

If m is even, n is odd, then the proof of this case is similar to case 2.

- Case 4: m, n are both even

If m, n are both even, then by equation (1.1), $\chi(C_m) = \chi(C_n) = 2$. Suppose the vertices of C_m and C_n are colored with the same set of different colors say a_1, a_2. Consider the following subcases where $k = 3$ (odd) and $k = 4$ (even):

- <u>subcase 4.1:</u> If $k = 3$

Then a path P_3 with vertices colored with the same set of different colors a_1, a_2, is attached to vertex colored say $a_i, i = 1, 2, 3$ of C_m and the other end vertex is also connected to a vertex colored a_i of C_n, such that the second (middle) vertex of P_3 is $a_j, j = 1, 2, 3$, where $a_i \neq a_j$. Thus, two is the minimum number of colors to color the vertices of $O^3{}_{n,m}$ where m is odd and n is even. Consequently, $\chi(O^3{}_{n,m}) = 2$.

- <u>subcase 4.2:</u> If $k = 4$

Note that by Remark 2.1, $\chi(P_4) = 2$. Suppose further that P_4 is colored with two from the same set of different colors that color the vertices of C_m and C_n, say a_1, a_2. Then, the first vertex of P_4 is attached to a vertex colored $a_i, i = 1,2$ of C_m and is adjacent to a vertex colored $a_j, j = 1,2$, which is the second vertex of P_4, and the second vertex is adjacent to a vertex colored $a_i, i = 1,2$, which is the third vertex of P_4, and the last vertex is then connected to a vertex colored $a_j, j = 1,2$ of C_n, where $a_i \neq a_j$. Thus, two is the minimum number of colors to color the vertices of $O^4{}_{n,m}$, where m is odd, n is even. Consequently, $\chi(O^4{}_{n,m}) = 2$. ■

It is easy to prove that the next corollaries hold. Proofs are similar to Theorem 3.1.

Corollary 3.1

For all integers $k \geq 3$, k is odd,

$$\chi(O^k{}_{n,m}) = 3 \qquad if:$$

- *m, n are both odd*
- *m is odd, n is even*
- *m is even, n is odd*

and

$$\chi(O^k{}_{n,m}) = 2 \qquad if\ m, n\ are\ both\ even.$$

Corollary 3.2 *For all integers $k \geq 2, k$ is even, $\chi(O^k{}_{n,m}) = 2$.*

Theorem 3.2 *For all integers $m, n \geq 3$ and for integers $k = 3, 4$, $\chi_e(O^k{}_{n,m}) = 3$.*

Proof:

- Case 1: m, n are both odd

If m, n are both odd, then by equation (1.2), $\chi_e(C_m) = \chi_e(C_n) = 3$. Thus, $\chi_e\left(O^k{}_{n,m}\right) \geq 3$. Suppose the edges of C_m and C_n are colored with the same set of different colors, say b_1, b_2, b_3.

- <u>subcase 1.1:</u> If $k = 3$

Note that P_3 has two edges and suppose we color its edges with two from the set of different colors that color the edges of C_m and C_n, say b_i, b_j. Then b_i color of P_3 is attached to C_m and is incident to edges colored b_j and b_k of C_m, while the other color of the edge of P_3 say b_j is attached to C_n and is incident to edges colored b_i and b_k of C_n, where $b_i \neq b_j \neq b_k$. Hence, three is the minimum number of colors that color the edges of $O^3{}_{n,m}$. Consequently, $\chi_e\left(O^3{}_{n,m}\right) = 3$.

- <u>subcase 2.1:</u> If $k = 4$

Note that by Remark 3.1, $\chi_e(P_4) = 2$ and suppose the edges of P_4 are colored with two from the same set of different colors that color the edges of C_m and C_n, say b_i, b_j. Since P_4 has three edges, suppose that P_4 is colored with $b_i's$ and b_j, $i = 1,2,3,\ j = 1,2,3$ such that b_j is the middle edge and the two $b_i's$ are first and the last edges. Then, b_i of P_4 is connected to C_m and is incident to edges colored b_j and b_k of C_m, while the other b_i of C_n. Hence, three is the minimum number of colors that color the edges of $O^4{}_{n,m}$. Consequently, $\chi_e\left(O^4{}_{n,m}\right) = 3$.

- Case 2: m is odd, n is even

If m is odd, n is even, then by equation (1.2), $\chi_e(C_m) = 3$ and $\chi_e(C_n) = 2$. Suppose the edges of C_n are colored with two from the same set of different colors that color the edges of C_m, say b_1, b_2 for C_n and b_1, b_2, b_3 for C_m. By this, the entire proof follows from case 1.

- Case 3: m is even, n is odd

If m is even, n is odd, then the proof of this case is similar to case 2.

- Case 4: m, n are both even

If m, n are both even, then by equation (1.2), $\chi_e(C_m) = \chi_e(C_n) = 2$. Suppose the edges of C_m are colored by a set of different colors say b_j, b_k and C_n is colored with the set of different colors, say b_i, b_k, where $i, j, k = 1, 2, 3$ and $b_i \neq b_j \neq b_k$.

- <u>subcase 4.1</u>: If $k = 3$

Note that by Remark 2.1, $\chi_e(P_3) = 2$. Clearly, $\chi_e(O^3{}_{n,m}) \geq 3$ since $O^3{}_{n,m}$ has three edges incident to each other at the endpoints of P_3. Then b_i color of P_3 is attached to C_m and is incident to edges colored b_j and b_k of C_m, while the other color of the edge of P_3 say b_j is attached to C_n and is incident to edges colored b_i and b_k of C_n and is incident to edges colored b_i and b_k of C_n, where $b_i \neq b_j \neq b_k$. Hence, three is the minimum number of colors that color the edges of $O^3{}_{n,m}$. Consequently, $\chi_e(O^3{}_{n,m}) = 3$.

- <u>subcase 4.2:</u> If $k = 4$

Note that P_4 has three edges and by Remark 3.1, $\chi_e(P_4) = 2$. Clearly, $\chi_e(O^4{}_{n,m}) \geq 3$ since $O^4{}_{n,m}$ has three edges incident to each other at the endpoints of P_k. Suppose that there are two $b_i's, i = 1,2,3$ and one $b_j, j = 1,2,3$ is the color of the middle edge, while the two $b_i's, i = 1,2,3$ are the colors of the first and the last edges. Then the first b_i color is attached to C_m and is incident to edges colored b_j and b_k colors C_m, while the other b_i is connected to C_n and is also incident to edges b_j and b_k colors of C_n. Hence, three is the minimum number of colors that color the edges of $O^4{}_{n,m}$. Consequently, $\chi_e(O^4{}_{n,m}) = 3$.

Therefore for any cases, the proof follows.

Similar arguments will hold for the last Theorem.

Theorem 3.3 For all integers $\mathrm{m, n} \geq 3$, and for all integers $\mathrm{k} \geq 3$, $\chi_e(\mathrm{O^k{}_{n,m}}) = 3$.

Corollary 3.3 For all integers $\mathrm{m, n} \geq 3$ and for any integers $\mathrm{k} \geq 3$,

$$\chi(O^k{}_{n,m}) \leq \chi_e(O^k{}_{n,m}) \leq 3.$$

Proof:

Note that for cases where m, n are both odd, m is odd, n is even and m is even, n is odd, by Theorem 3.1, $\chi(O^k{}_{n,m}) = 3$ and by Theorem 3.2, $\chi_e(O^k{}_{n,m}) = 3$. Thus, $\chi(O^k{}_{n,m}) \leq \chi_e(O^k{}_{n,m})$. Similarly, for cases where m, n are both even, by Theorem 3.1, $\chi(O^k{}_{n,m}) = 2$ and by Theorem 3.2, $\chi_e(O^k{}_{n,m}) = 3$. Thus, $\chi(O^k{}_{n,m}) \leq \chi_e(O^k{}_{n,m})$. Therefore, in all cases, $\chi(O^k{}_{n,m}) \leq \chi_e(O^k{}_{n,m})$.

Acknowledgements

The authors would like to thank the referee for the helpful suggestions and the members of ASRJETS. Special thanks to authors' families and friends.

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